Time scales of relaxation and Lyapunov instabilities in a one-dimensional gravitating sheet system

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Abstract

The relation between relaxation, the time scale of Lyapunov instabilities, and the Kolmogorov-Sinai time in a one-dimensional gravitating sheet system is studied. Both the maximum Lyapunov exponent and the Kolmogorov-Sinai entropy decrease as proportional to $N^{-1/5}$. The time scales determined by these quantities evidently differ from any type of relaxation time found in the previous investigations. The relaxation time to quasiequilibria (microscopic relaxation) is found to coincide with the inverse of the minimum positive Lyapunov exponent. The relaxation time to the final thermal equilibrium differs to the inverse of the Lyapunov exponents and the Kolmogorov-Sinai time.

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I. INTRODUCTION

Relaxation is the most fundamental process in evolution of many-body system. The classical statistical theory is based on ergodic property, which is considered to be established after relaxation. However, not all systems do not show such an idealistic relaxation. A historical example is FPU (Fermi-Pasta-Ulam) problem [1], which experiences the induction phenomenon (e.g., Ref [2,3]) and does not relax to the equipartition for very long time.

From nearly thirty years of investigation, one-dimensional self-gravitating sheet systems (OGS) have been known by their strange behavior in evolution. Hohl [4–6] first asserted that OGS relaxes to the thermodynamical equilibrium (the isothermal distribution) in a time scale of about $N^2 t_c$, where N is the number of sheets, and t_c is typical time for a sheet to cross the system. Later, more precise numerical experiments determined that the Hohl's result was not right, and then arguments for the relaxation time arose in 1980's. A Belgian group [7,8] claimed the OGS relaxed in shorter than $N t_c$, whereas a Texas group [9,10] showed that the system showed long lived correlation and never relaxed even after $2N^2 t_c$. Tsuchiya, Gouda and Konishi (1996) [11] suggested that this contradiction can be resolved in the view of two different types of relaxations: the *microscopic* and the *macroscopic* relaxations. At the time scale of $N t_c$, cumulative effect of the mean field fluctuation makes the energies of the individual particles change noticeably. Averaging this change gives the equipartition of energies, thus there is a relaxation at this time scale. By this relaxation the system is led not to the thermal equilibrium but only to a quasiequilibrium. The global shape of the one-body distribution remains different from that of the thermal equilibrium. This relaxation appears only in the microscopic dynamics, thus it is called the microscopic relaxation. The global shape of the one-body distribution transforms in much longer time scale. For example, a quasiequilibrium (the water-bag distribution, which has the longest life time) begins to transform at $4 \times 10^4 N t_c$ in average. Tsuchiya et al.(1996) [11] called this transformation the macroscopic relaxation, but later in Tsuchiya, Gouda and Konishi (1998) [12], it is shown that this transformation is onset of the *itinerant stage*. In this stage, the

one-body distribution stays in a quasiequilibrium for some time and then changes to other quasiequilibrium. This transformation continues forever. Probability density of the life time of the quasiequilibria has a power law distribution with a long time cut-off and the longest life time is $sim10^4 Nt_c$. Only by averaging over a time longer than the longest life time of the quasiequilibria, the one-body distribution becomes that of the thermal equilibrium, which is defined as the maximum entropy state. Therefore the time $\sim 10^6 Nt_c$ is necessary for relaxation to the thermal equilibrium, and called the *thermal relaxation time.* Although there are some attempts to clarify the mechanisms of these relaxations [11,13,14,12], the reason why the system does not relax for such a long time is still unclear.

At the view of chaotic theory of dynamical systems, relaxation is understood as mixing in phase space, and its time scale is given by the Kolmogorov-Sinai time (KS time), $\tau_{\mathrm{KS}} = 1/h_{\mathrm{KS}}$, where h_{KS} is the Kolmogorov-Sinai entropy. However, it does not simply correspond to the relaxation of the one-body distribution function, which is of interest in many-body systems. Recently, Dellago and Posch [15] showed that in a hard sphere gas, the KS time equals the mixing time of neighboring orbits in the phase space, whereas the relaxation of the one-body distribution function corresponds to the collision time between particles. Now, it is fruitful to study relation between relaxation and some dynamical quantities, such as the KS entropy and the Lyapunov exponents, in the OGS. Milanović et al. [14] showed the Lyapunov spectrum and the Kolmogorov-Sinai entropy in the OGS for $10 \leq N \leq 24$. However, since it is known that the chaotic behavior changes for $N \sim 30$ for the OGS [16], it is considerably important to extend the analysis to the system larger than $N \sim 30$. In this paper, we extend the number of sheets to $N = 256$ and follow the evolution numerically up to $T \sim 10^6 Nt_c$, which is long enough for the thermal relaxation [12].

II. NUMERICAL SIMULATIONS

The OGS comprises N identical plane-parallel mass sheets, each of which has uniform mass density and infinite in, say, the y and z direction. They move only in the x direction

under their mutual gravity. When two of the sheets intersect, they pass through each other. The Hamiltonian of the system has the form

$$H = \frac{m}{2} \sum_{i=1}^{N} v_i^2 + (2\pi G m^2) \sum_{i<j} |x_j - x_i|, \tag{1}$$

where m, v_i, and x_i are the mass (surface density), velocity, and position of the ith sheet, respectively. Since the gravitational field is uniform, the individual particles moves parabolically, until they intersect with the neighbors. Thus the evolution of the system can be followed by solving quadratic equations. This property helps us to calculate long time evolution with a high accuracy. Since length and velocity (thus also energy) can be scaled in the system, the number of the sheets N is the only free parameter. The crossing time is defined by

$$t_c = (1/4\pi GM)(4E/M)^{1/2}, \tag{2}$$

where M and E is the total mass and total energy of the system. Detailed descriptions of the evolution of the OGS can be found in our previous papers [17,11,12].

In order to investigate dynamical aspects of the system, we calculated the Lyapunov spectrum. The basic numerical algorithm follows Shimada and Nagashima [18], and detailed description of the procedure for the OGS can be found in ref [17,14]. We made numerical integration for $8 \leq N \leq 128$ up to $10^8 t_c$, which is enough time for the system to relax, and up to $1.8 \times 10^7 t_c$ for $N = 256$ for reference.

III. RESULTS

Figure 1 shows the spectrum of the Lyapunov exponents, $\{\lambda_i\}$, where their unit is $1/t_c$. This figure is the same diagram as Fig. 6 in Milanović et al [14], but the range of N is extended to $8 \leq N \leq 256$. In the horizontal axis, l is the index of the Lyapunov exponents, which is labeled in the order from the maximum to the minimum. Thus all the positive Lyapunov exponents ($l \leq N$) is scaled between 0 to 1 in the axis. The vertical axis shows the

Lyapunov exponents normalized by the maximum Lyapunov exponents, λ_1. Milanović et al [14] stated that the shape of the spectrum approximately converges for large N. A closer look, however, shows bending of the spectrum, which is most clearly seen at $(N-l)/(N-1) \sim 0.9$. This bending seems increase with N for $N \geq 32$. A further investigation is needed to give a definite conclusion about the convergence of the shape of the spectrum.

Figure 2 shows N-dependence of the maximum (λ_1), the minimum positive Lyapunov exponent (λ_{N-2}), and the KS entropy h_{KS} per the number of freedom. λ_1 is already shown in Fig.13 in Tsuchiya et al. [17], and it is proportional to $N^{-1/5}$ for $N \geq 32$. Decreasing nature of the Lyapunov exponent may indicate that the OGS approaches closer to an integrable system for larger N.

As expected from the spectrum the KS entropy divided by N is also proportional to $N^{-1/5}$. Therefore the conjecture by Benettin et al. [19] that h_{KS} increases linearly with N is not right. It is clear that the inverses of both the maximum Lyapunove exponents and the KS entropy do not give the time scale of any type of relaxation time.

The N-dependence of small positive Lyapunov exponents are quite different from larger ones. In Fig.2, the minimum positive Lyapunov exponent, λ_{N-2}, is shown by a dashed dotted line with the symbol $\triangle$. It decreases linearly for $N \geq 32$, and its time scale $1/\lambda_{N-2}$ is about the same as the microscopic relaxation time $(\sim Nt_c)$.

The eigen vectors for the Lyapunov exponents also give a useful information. Figure 3 shows projection of the eigen vector for $N = 64$ on to the one-body phase space. Filled circles indicate positions of N sheets at a moment and the arrows give the direction of the Lyapunov eigen vector, which grows with the rate of the Lyapunov exponent. Fig 3(a) is for the maximum Lyapunov exponent λ_1, and Fig.3(b) is for the minimum positive one, λ_{N-2}. For λ_1, the instability is carried only by a few particles, which are interacting in a very small region. The instability is thus not for global transformation. On the other hand, the instability with λ_{N-2} makes all particles mix in the phase space. This is the very effect of relaxation. These features are commonly seen for different N.

The results that the coincidence of the $1/\lambda_{N-2}$ and the microscopic relaxation time, and

the direction of the eigen vector, may be suggesting that the microscopic relaxation time is determined by the growing time of the weakest instability, which is determined by the minimum positive Lyapunov exponent; in other words, this time is necessary for the phase space orbit to mix in the phase space in the all directions of freedom. In our working model of the evolution of the OGS [11,12], the phase space is derived by some barriers which keep the phase orbit inside for a long time. The microscopic relaxation is considered to be a diffusion process in the barierred region [11,13], and in the time $\sim N t_c$, restricted ergodicity is established within the barierred region. This time may correspond to the diffusion time in the slowest direction.

IV. CONCLUSIONS AND DISCUSSION

In the ergodic theory, the KS time represents the time scale of "Mixing" in the phase space. On the other hand, the relaxation of the one-body distribution is of the most interest in systems with large degrees of freedom. We have shown that the time scale of the relaxation of one-body distribution (both the microscopic and thermal relaxation) is certainly different from that of the KS time, and found that the growing time of the weakest Lyapunov instability is about the same as the microscopic relaxation time. In addition, taking into account the direction of the eigen vector of the weakest Lyapunov exponent, it is suggested that the microscopic relaxation is determined by the weakest Lyapunov instability.

The KS entropy is defined as a typical time for the system to increase "information". This definition does not depend on the number of degrees of freedom. In higher dimensions, however, even very small growth of instability can increase information quite rapidly. Therefore the KS time does not seem suitable to characterize the relaxation of the one-body distribution function.

The relaxation of the one-body distribution function implies ergodicity. To attain ergodicity, the phase space orbits should diffuse over all accessible phase space. For the microscopic relaxation, even though it is not true thermal relaxation, the system shows er-

godicity which is restricted in a part of the phase space [11]. Therefore it seems natural that the microscopic relaxation time is characterized by the slowest time of diffusion. That may be the reason that the inverse of the minimum positive Lyapunov exponent coincides the microscopic relaxation time.

Gurzadyan and Savvidy [20] derived the KS time in the usual three dimensional stellar systems, which was found to be proportional to $N^{1/3}$, by the method of the geodesic deviation. They asserted that the system relaxes in this time scale. This result was not supported by numerical simulations [21] and by a semi-analytical study [22]. Our analysis of one-dimensional systems also gives negative result for the conjecture of Gurzadyan and Savvidy. According to our results, the time scale of the minimum positive Lyapunov exponent would be related to the relaxation also in three-dimensional systems, though it is difficult to show it numerically.

We have found that the KS time and any of the times of Lyapunov instabilities do not give the thermal relaxation time in the OGS. In our working model, the thermal relaxation is the successive transitions of the phase space orbit among the barierred regions. The each region has locally restricted ergodicity, hence the long time average could only give average of the Lyapunov exponents, which are defined in each region, and it is reasonable that the averaged Lyapunov exponents do not characterize the thermal relaxation. Actual time of the thermal relaxation is that of transition among quasiequilibria. It is necessary to find appropriate dynamical quantities to describe it.

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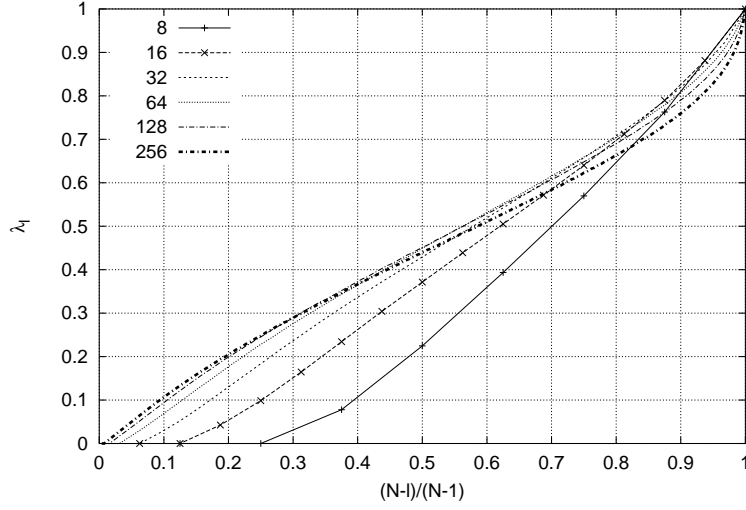[22] J. Goodman, D. C. Heggie, and P. Hut, Astrophys. J. **415**, 715 (1993).

FIG. 1. Spectrum of the positive Lyapunov exponents for various N. The index of the Lyapunov exponents is scaled to 0 to 1.0. The vertical axis shows the Lyapunov exponents normalized by the value of the maximum Lyapunov exponent.
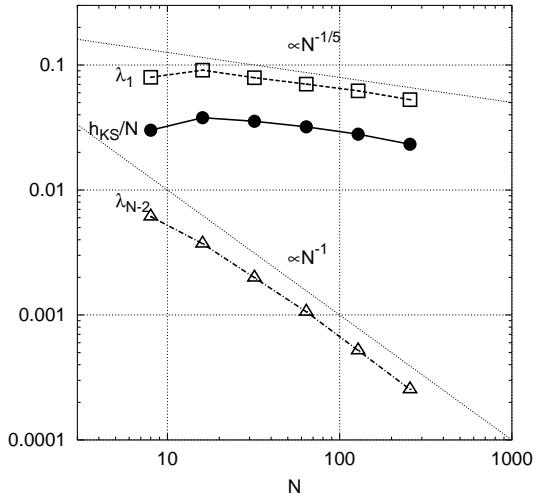

FIG. 2. Dependence of the KS entropy (solid line with the symbol $\bullet$), the maximum Lyapunov exponent (long dashed curve with the symbol $\square$), and the minimum positive Lyapunov exponent (dashed dotted curve with the symbol $\triangle$).
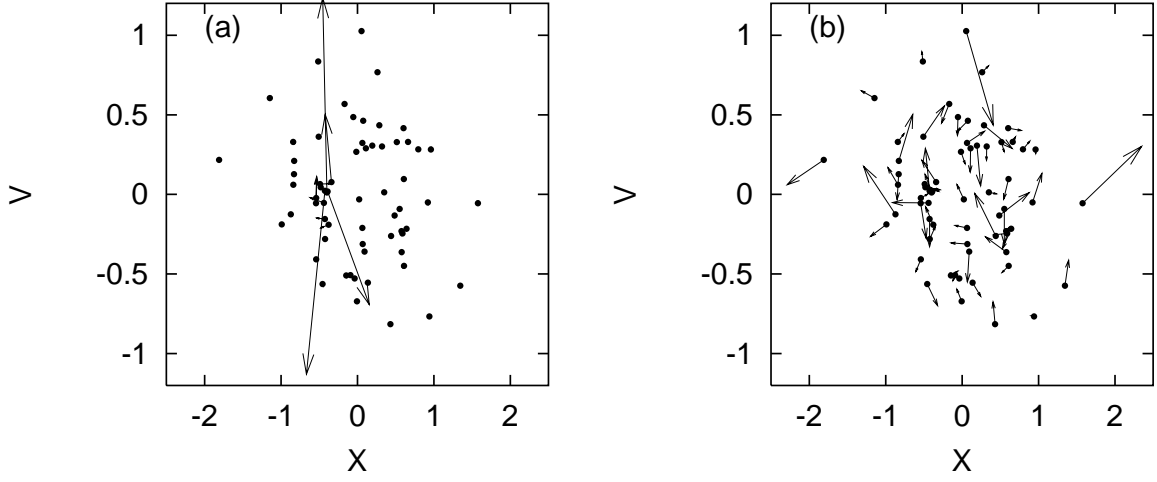
FIG. 3. Lyapunov eigen vectors for $N = 64$. Filled circles indicate positions of N sheets and the arrows give direction of the Lyapunov eigen vector: (a) the eigen vector for λ_1, (b) that for λ_{N-2}.